Gradient, Divergence and curl.

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* The Del operator (∇ or $\vec{\nabla}$) :-

The Del operator $\vec{\nabla}$ is defined as

$$\vec{\nabla} = \hat{x} \cdot \frac{\delta}{\delta x} + \hat{y} \cdot \frac{\delta}{\delta y} + \hat{z} \cdot \frac{\delta}{\delta z}$$

or $$\vec{\nabla} = \hat{i} \frac{\delta}{\delta x} + \hat{j} \frac{\delta}{\delta y} + \hat{k} \cdot \frac{\delta}{\delta z} .$$

Del ($\vec{\nabla}$) is not a vector but it is a vector operator.

* Del of T ($\vec{\nabla} T$) is a vector where T is scalar.

$$\vec{\nabla} T = \left(\hat{x} \frac{\delta}{\delta x} + \hat{y} \frac{\delta}{\delta y} + \hat{z} \frac{\delta}{\delta z} \right) T$$

* $$dT = \frac{\delta T}{\delta x} \cdot dx + \frac{\delta T}{\delta y} \cdot dy + \frac{\delta T}{\delta z} \cdot dz$$

or $$dT = \left(\frac{\delta T}{\delta x} \cdot \hat{x} + \frac{\delta T}{\delta y} \cdot \hat{y} + \frac{\delta T}{\delta z} \cdot \hat{z} \right) \cdot \left(dx \cdot \hat{x} + dy \cdot \hat{y} + dz \cdot \hat{z} \right)$$

or $$dT = \left(\hat{x} \cdot \frac{\delta}{\delta x} + \hat{y} \frac{\delta}{\delta y} + \hat{z} \frac{\delta}{\delta z} \right) T \cdot \left(\hat{x}\, dx + \hat{y}\, dy + \hat{z}\, dz \right)$$

or $$dT = (\vec{\nabla} T) \cdot (d\vec{r})$$

where $\vec{\nabla} T = \hat{x} \frac{\delta T}{\delta x} + \hat{y} \frac{\delta T}{\delta y} + \hat{z} \frac{\delta T}{\delta z}$

is known as gradient of T

and $d\vec{r} = \hat{x} \cdot dx + \hat{y} \cdot dy + \hat{z} \cdot dz$ is distance.

* Gradient is used on scalar function.

$\vec{\nabla} T$ = Gradient of T (Grad T) where T = scalar.

Example :1 Find gradient of the following functions.

(a) $f(x,y,z) = x^2+y^3+z^4$ (b) $f(x,y,z) = x^2y^3z^4$

(c) $f(x,y,z) = e^x.\sin y.\ln z$.

Ans:- (a) $f(x,y,z) = x^2+y^3+z^4$.

Gradient of the function $f(x,y,z)$ is

$$\vec{\nabla} f = \hat{x}\frac{\delta f}{\delta x} + \hat{y}.\frac{\delta f}{\delta y} + \hat{z}.\frac{\delta f}{\delta z}$$

$$= \hat{x}.2x + \hat{y}.3y^2 + \hat{z}.4z^3$$

(b) $f(x,y,z) = x^2y^3z^4$.

Gradient of the function $f(x,y,z)$ is

$$\vec{\nabla} f = \hat{x}\frac{\delta f}{\delta x} + \hat{y}\frac{\delta f}{\delta y} + \hat{z}\frac{\delta f}{\delta z}$$

$$= \hat{x}.2xy^3z^4 + \hat{y}.3x^2y^2z^4 + \hat{z}.4x^2y^3z^3.$$

(c) $f(x,y,z) = e^x.\sin y.\ln z$.

Gradient of the function $f(x,y,z)$ is

$$\vec{\nabla} f = \hat{x}.\frac{\delta f}{\delta x} + \hat{y}\frac{\delta f}{\delta y} + \hat{z}\frac{\delta f}{\delta z}$$

$$= \hat{x}.e^x.\sin y \ln z + \hat{y}.e^x.\cos y.\ln z + \hat{z}.e^x\sin y.\frac{1}{z}.$$

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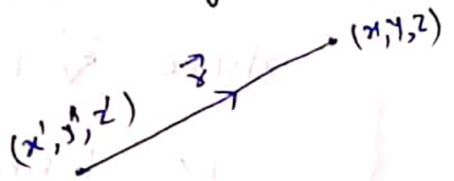
Example: 2 Let $\vec{r}$ be the separation vector from a fixed point (x',y',z') to the point (x,y,z) and r be its length. Show that

(a) $\vec{\nabla}(r^2) = 2\vec{r}$ (b) $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$ (c) what is the general formula for $\vec{\nabla}(r^n)$?



Ans:- $\vec{r} = (x-x')\hat{x} + (y-y').\hat{y} + (z-z')\hat{z}$

$$r = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

or $r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$.

(a) $$\vec{\nabla}(r^2) = \hat{x}.\frac{\delta r^2}{\delta x} + \hat{y}\frac{\delta r^2}{\delta y} + \hat{z}.\frac{\delta r^2}{\delta z}.$$

$$= 2(x-x').\hat{x} + 2(y-y').\hat{y} + 2(z-z').\hat{z}$$

$\vec{\nabla}(r^2) = 2.\vec{r}$. proved.

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(b) $\vec{\nabla}\left(\frac{1}{r}\right) = \frac{\delta\left(\frac{1}{r}\right)}{\delta x}\cdot\hat{x} + \frac{\delta\left(\frac{1}{r}\right)}{\delta y}\cdot\hat{y} + \frac{\delta\left(\frac{1}{r}\right)}{\delta z}\cdot\hat{z}$

$$= -\frac{1}{r^2}\cdot\frac{\delta r}{\delta x}\cdot\hat{x} + \left(-\frac{1}{r^2}\right)\cdot\frac{\delta r}{\delta y}\cdot\hat{y} + \left(-\frac{1}{r^2}\right)\frac{\delta r}{\delta z}\cdot\hat{z}$$

$$= -\frac{1}{r^2}\cdot\frac{1\times 2(x-x')}{2\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\cdot\hat{x} - \frac{1}{r^2}\frac{1\times 2(y-y')}{2\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\cdot\hat{y}$$

$$-\frac{1}{r^2}\cdot\frac{1\times 2(z-z')}{2\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\cdot\hat{z}$$

$$= -\frac{1}{r^2}\cdot\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\left[(x-x')\cdot\hat{x}+(y-y')\cdot\hat{y} + (z-z')\cdot\hat{z}\right]$$

$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\cdot\frac{\vec{r}}{r} = -\frac{\hat{r}}{r^2}$ proved $\quad \because \hat{r} = \frac{\vec{r}}{r}$.

(c) $\vec{\nabla}(r^n) = \frac{\delta r^n}{\delta x}\cdot\hat{x} + \frac{\delta r^n}{\delta y}\cdot\hat{y} + \frac{\delta r^n}{\delta z}\cdot\hat{z}$

$$= n\cdot r^{n-1}\cdot\frac{\delta r}{\delta x}\cdot\hat{x} + n\cdot r^{n-1}\cdot\frac{\delta r}{\delta y}\cdot\hat{y} + n\cdot r^{n-1}\cdot\frac{\delta r}{\delta z}\cdot\hat{z}$$

$$= n\cdot r^{n-1}\cdot\frac{2(x-x')\cdot\hat{x} + 2(y-y')\cdot\hat{y} + 2(z-z')\cdot\hat{z}}{2\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$= n\cdot r^{n-1}\cdot\frac{(x-x')\cdot\hat{x} + (y-y')\cdot\hat{y} + (z-z')\cdot\hat{z}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$= n\cdot r^{n-1}\cdot\frac{\vec{r}}{r}$$

$$\Rightarrow \vec{\nabla}(r^n) = n\cdot r^{n-1}\cdot\hat{r}.$$

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The Divergence : Divergence of a vector:

If $\vec{F}(x,y,z)$ is a vector function the divergence of the vector function $\vec{F}$ is denoted by $\vec{\nabla}\cdot\vec{F}$ and it is defined as

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F}$$

where $\vec{\nabla} = \frac{\delta}{\delta x}\cdot\hat{x} + \frac{\delta}{\delta y}\cdot\hat{y} + \frac{\delta}{\delta z}\cdot\hat{z}$ and $\vec{F} = F_x\hat{x} + F_y\cdot\hat{y} + F_z\cdot\hat{z}$

Now $\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\frac{\delta}{\delta x}\cdot\hat{x} + \frac{\delta}{\delta y}\cdot\hat{y} + \frac{\delta}{\delta z}\cdot\hat{z}\right)\cdot\left(F_x\hat{x} + F_y\cdot\hat{y} + F_z\cdot\hat{z}\right)$

$$\Rightarrow \vec{\nabla}\cdot\vec{F} = \frac{\delta F_x}{\delta x} + \frac{\delta F_y}{\delta y} + \frac{\delta F_z}{\delta z}.$$

* Divergence of a vector function $\vec{F}$ gives scalar function.
* Divergence of a vector function $\vec{F}$ is equal to dot product of the del vector ($\vec{\nabla}$) and the vector function $\vec{F}$.
* Divergence is used on vector function only via dot product.

Geometrical Meaning or physical significance of Divergence

Divergence of a vector field at a point means how much field is leaving (spreading out or diverging) from that point.

Consider an infinitesimal (a very small) volume around the point at which we are going to find the divergence of the vector field. If we find that some field is entering that region and some is leaving the small region then net field going out the small region is considered as the divergence of the field at that point. If field lines entering the point is equal to field lines leaving the point the divergence of the field at that point will be zero.

Example:3 Calculate divergence of the following vector functions.

(1) $\vec{F} = x\cdot\hat{x} + y\cdot\hat{y} + z\cdot\hat{z}$ (2) $\vec{F} = \hat{z}$ (3) $\vec{F} = z\cdot\hat{z}$
(4) $\vec{F} = x^2\cdot\hat{x} + 3xz^2\hat{y} + 2xy\cdot\hat{z}$ (5) $\vec{F} = x\cdot\hat{x} - y^2\cdot\hat{y} - z^3\cdot\hat{z}$.

Ans:- (1) $\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\frac{\delta}{\delta x}\cdot\hat{x} + \frac{\delta}{\delta y}\cdot\hat{y} + \frac{\delta}{\delta z}\cdot\hat{z}\right)\cdot\left(x\cdot\hat{x} + y\cdot\hat{y} + z\cdot\hat{z}\right)$

$= 1 + 1 + 1 = 3.$

(2) $\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\frac{\delta}{\delta x}\hat{x} + \frac{\delta}{\delta y}\cdot\hat{y} + \frac{\delta}{\delta z}\cdot\hat{z}\right)\cdot(\hat{z})$

$$= 0 + 0 + \frac{\delta 1}{\delta z} = 0$$

(3) $\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\frac{\delta}{\delta x}\cdot\hat{x} + \frac{\delta}{\delta y}\cdot\hat{y} + \frac{\delta}{\delta z}\cdot\hat{z}\right)\cdot(z\cdot\hat{z})$

$$= 0 + 0 + \frac{\delta z}{\delta z} = 1.$$

(4) $\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\frac{\delta}{\delta x}\hat{x} + \frac{\delta}{\delta y}\cdot\hat{y} + \frac{\delta}{\delta z}\cdot\hat{z}\right)\cdot(x^2\cdot\hat{x} + 3xz^2\cdot\hat{y} + 2xy\,\hat{z})$

$$= \frac{\delta x^2}{\delta x} + \frac{\delta(3xz^2)}{\delta y} + \frac{\delta(2xy)}{\delta z}$$

$$= 2x + 0 + 0 = 2x.$$

(5) $\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\frac{\delta}{\delta x}\cdot\hat{x} + \frac{\delta}{\delta y}\cdot\hat{y} + \frac{\delta}{\delta z}\cdot\hat{z}\right)\cdot(x\cdot\hat{x} - y^2\cdot\hat{y} - z^3\cdot\hat{z})$

$$= \frac{\delta x}{\delta x} - \frac{\delta y^2}{\delta y} - \frac{\delta z^3}{\delta z} = 1 - 2y - 3z^2.$$

The curl : Curl of a vector function :

If $\vec{F}(x,y,z)$ is a vector function then the curl of the vector function $\vec{F}$ is denoted by $\vec{\nabla}\times\vec{F}$ and it is defined as

$$\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F}$$

where $\vec{\nabla} = \frac{\delta}{\delta x}\cdot\hat{x} + \frac{\delta}{\delta y}\cdot\hat{y} + \frac{\delta}{\delta z}\cdot\hat{z}$ and $\vec{F} = F_x\cdot\hat{x} + F_y\cdot\hat{y} + F_z\cdot\hat{z}$

Now $\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \left(\frac{\delta}{\delta x}\cdot\hat{x} + \frac{\delta}{\delta y}\cdot\hat{y} + \frac{\delta}{\delta z}\cdot\hat{z}\right)\times(F_x\hat{x} + F_y\cdot\hat{y} + F_z\hat{z})$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\Rightarrow \text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \hat{x}\left(\frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z}\right) - \hat{y}\left(\frac{\delta F_z}{\delta x} - \frac{\delta F_x}{\delta z}\right) + \hat{z}\left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right)$$

* curl is used on vector function only via cross product.
* curl of a vector function $\vec{F}$ is equal to cross product of the del operator $(\vec{\nabla})$ and the vector function $\vec{F}$.
* curl of a vector function $\vec{F}$ gives vector function.

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Geometrical Meaning or physical significance of curl

Curl means rotation. Curl of a vector field at a point means how much the field is rotating about the point or how much the field has rotating effect at that point. The direction of curl is the axis of rotation as determined by the right hand screw rule and magnitude of curl is magnitude of rotation. A vector field whose curl is zero, is known as Irrotational. If curl $\vec{F} = 0$ then the field $\vec{F}$ is irrotational but if curl $\vec{F} \neq 0$ then the field $\vec{F}$ is rotational. Curl $\vec{F}$ is a measure of how much the vector $\vec{F}$ swirls (rotates) around the point.

Example :- Calculate curl of the following vector functions.

(1) $\vec{F} = x^2 \cdot \hat{x} + 3xz^2 \cdot \hat{y} - 2xz \cdot \hat{z}$ (2) $\vec{F} = xy \cdot \hat{x} + 2yz \cdot \hat{y} + 3zx \cdot \hat{z}$.

Ans :- (1) curl of the vector function $\vec{F}$ is

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix}$$

$$= \hat{x}\left\{\frac{\delta(-2xz)}{\delta y} - \frac{\delta 3xz^2}{\delta z}\right\} - \hat{y}\left\{\frac{\delta(-2xz)}{\delta x} - \frac{\delta x^2}{\delta z}\right\} + \hat{z}\left\{\frac{\delta 3xz^2}{\delta x} - \frac{\delta x^2}{\delta y}\right\}$$

$$= \hat{x}(0 - 6xz) - \hat{y}(-2z - 0) + \hat{z}(3z^2 - 0)$$

$$\vec{\nabla} \times \vec{F} = -6xz \cdot \hat{x} + 2z \cdot \hat{y} + 3z^2 \cdot \hat{z}.$$

(2) curl of the vector function $\vec{F}$ is

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ xy & 2yz & 3zx \end{vmatrix}$$

$$= \hat{x}\left\{\frac{\delta 3zx}{\delta y} - \frac{\delta 2yz}{\delta z}\right\} - \hat{y}\left\{\frac{\delta 3zx}{\delta x} - \frac{\delta xy}{\delta z}\right\} + \hat{z}\left\{\frac{\delta 2yz}{\delta x} - \frac{\delta xy}{\delta y}\right\}$$

$$= \hat{x}(0 - 2y) - \hat{y}(3z - 0) + \hat{z}(0 - x)$$

$$\Rightarrow \vec{\nabla} \times \vec{F} = -2y \cdot \hat{x} - 3z \cdot \hat{y} - x \cdot \hat{z}$$

———— x ————

Ques:- What do you mean by gradient of a scalar, divergence and curl of a vector? Explain their physical significance. Show that curl of linear velocity of a particle is twice its angular velocity.

Ans:- Gradient of a scalar:- If $\phi(x,y,z)$ is a scalar function then gradient of the scalar function ϕ is denoted by $\nabla\phi$ and it is defined as

$$\text{grad}\,\phi = \nabla\phi = \left(\frac{\delta}{\delta x}\cdot\hat{i} + \frac{\delta}{\delta y}\cdot\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\phi$$

$$\Rightarrow \text{grad}\,\phi = \nabla\phi = \frac{\delta\phi}{\delta x}\cdot\hat{i} + \frac{\delta\phi}{\delta y}\cdot\hat{j} + \frac{\delta\phi}{\delta z}\cdot\hat{k}$$

* Gradient is used on scalar function only.

⊕ Divergence of a vector:- If $\vec{F}(x,y,z)$ is a vector function then divergence of the vector function $\vec{F}$ is denoted by $\vec{\nabla}\cdot\vec{F}$ and it is defined as

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F}$$

where $\vec{\nabla} = \frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}$ and $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\cdot\hat{k}$

Now $\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\cdot\left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right)$

$$\Rightarrow \text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\delta F_x}{\delta x} + \frac{\delta F_y}{\delta y} + \frac{\delta F_z}{\delta z}.$$

* Divergence of vector function $\vec{F}$ gives scalar function.
* Divergence of a vector function $\vec{F}$ is equal to dot product of del operator (∇) and the vector function $\vec{F}$.
* Divergence is used on vector function only via dot product.

Geometrical meaning or physical significance of divergence:
Divergence of a vector field at a point means how much field is leaving (spreading out) from that point.

Consider an infinitesimal (a very small) volume around the point at which we are going to find the divergence of the vector field. If we find that some field is entering that small region and some is leaving. The net field going out the region is considered as the divergence of the field at that point. If field lines entering the point is

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equal to field lines leaving the point then divergence of the field at that point will be zero.

⇔ Curl of a vector:- If $\vec{F}(x,y,z)$ is a vector function then curl of the vector function $\vec{F}$ is denoted by $\vec{\nabla}\times\vec{F}$ and it is defined as

$$\text{Curl } \vec{F} = \vec{\nabla}\times\vec{F}$$

where $\vec{\nabla} = \frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}$ and $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$

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Now
$$\text{Curl } \vec{F} = \vec{\nabla}\times\vec{F}$$
$$= \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\times\left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right)$$

$$\Rightarrow \text{Curl } \vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\Rightarrow \text{Curl } \vec{F} = \vec{\nabla}\times\vec{F} = \hat{i}\left(\frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z}\right) - \hat{j}\left(\frac{\delta F_z}{\delta x} - \frac{\delta F_x}{\delta z}\right) + \hat{k}\left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right)$$

* Curl is used on vector function only via cross product.
* Curl of a vector function $\vec{F}$ is equal to cross product of del operator (∇) and the vector function $\vec{F}$.
* Curl of a vector function $\vec{F}$ gives vector function.

Geometrical Meaning or physical significance of curl:

Curl means rotation. The curl of a vector function at a point tells us how much the field is rotating or has the rotating effect. The direction of curl is the axis of rotation as determined by the right hand rule and magnitude of curl is the magnitude of rotation.

A vector field whose curl is zero, is called irrotational. If curl $\vec{F} = 0$ then the field $\vec{F}$ is irrotational but if curl $\vec{F} \neq 0$ then the field $\vec{F}$ is rotational. Curl $\vec{F}$ is a measure of how much the vector $\vec{F}$ swirls ~~around~~ (rotates) around the point.

We know that $\vec{v} = \vec{\omega} \times \vec{r}$

where $\vec{\omega}$ = angular velocity, $\vec{v}$ = linear velocity and
$\vec{r}$ = position vector of a point on the rotating body.

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since $\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

so $\vec{V} = \vec{\omega} \times \vec{r} = (\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) \times (x\hat{i} + y\hat{j} + z\hat{k})$

$$\Rightarrow \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$\Rightarrow \vec{V} = \hat{i}(\omega_2 z - \omega_3 y) - \hat{j}(\omega_1 z - \omega_3 x) + \hat{k}(\omega_1 y - \omega_2 x)$$

Now $\text{curl}\, \vec{V} = \vec{\nabla} \times \vec{V}$

$$= \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right) \times \left[\hat{i}(\omega_2 z - \omega_3 y) + \hat{j}(\omega_3 x - \omega_1 z) + \hat{k}(\omega_1 y - \omega_2 x)\right]$$

$$\text{Curl}\, \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\delta}{\delta z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \hat{i}\left(\frac{\delta(\omega_1 y - \omega_2 x)}{\delta y} - \frac{\delta(\omega_3 x - \omega_1 z)}{\delta z}\right] - \hat{j}\left[\frac{\delta(\omega_1 y - \omega_2 x)}{\delta x} - \frac{\delta(\omega_2 z - \omega_3 y)}{\delta z}\right]$$

$$+ \hat{k}\left[\frac{\delta(\omega_3 x - \omega_1 z)}{\delta x} - \frac{\delta(\omega_2 z - \omega_3 y)}{\delta y}\right]$$

$$= \hat{i}[\omega_1 + \omega_1] - \hat{j}(-\omega_2 - \omega_2) + \hat{k}(\omega_3 + \omega_3]$$

$$= 2(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k})$$

$$\text{curl}\, \vec{v} = 2\vec{\omega}$$

Therefore curl of linear velocity ($\vec{v}$) of a particle is twice its angular velocity. proved.

Short notes on Scalar and vector fields

A physical quantity which can be expressed as a continuous function of the position of the point in a region of space, is known as point or position function.

Field :- Field is a function that specifies a particular physical quantity every where in a region. If the quantity is scalar then the field is known as scalar field and if the quantity is vector then the field is known as vector field.

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* Scalar field :- A scalar function ϕ which has values throughout a region (every where in a region) of space constitutes a field, known as scalar field.

Example :- Temperature distribution in a block or building, Sound intensity in a hall, electric or gravitational potential in a region of electric or gravitational field, Refractive index of a medium etc are examples of scalar fields.

* Vector field : A vector function $\vec{F}$ which has values throughout a region (every where in a region) of space constitutes a field, known as vector field.

Example :- velocity field, force field (electric force field, gravitational force field, magnetic force field) etc are examples of vector field.

Gradient is applied on scalar field only.

$$\text{grad}\,\phi = \nabla\phi = \frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}$$

Divergence and curl are applied on vector field only.

Divergence is applied on vector field $\vec{F}$ via dot product as

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\cdot\left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right)$$

$$\Rightarrow \text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\delta F_x}{\delta x} + \frac{\delta F_y}{\delta y} + \frac{\delta F_z}{\delta z}.$$

Therefore divergence of vector field $\vec{F}$ gives a scalar function

Curl is applied on vector field $\vec{F}$ via cross product as

$$\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\times\left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_x & F_y & F_z \end{vmatrix}.$$

$$\Rightarrow \text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \hat{i}\left(\frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z}\right) - \hat{j}\left(\frac{\delta F_z}{\delta x} - \frac{\delta F_x}{\delta z}\right) + \hat{k}\left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right)$$

Therefore, curl of vector field $\vec{F}$ gives a vector function

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